ON THE PROPERTIES OF THE RELATIONS OF THE LAW OF ANISOTROPIC HARDENING OF PLASTIC MATERIAL

(O SVOISTVAKH SOOTNOSHENII ZAKONA ANIZOTROPNOGO UPROCHNENIIA PLASTICHESKOGO MATERIALA)

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Ishlinskii [1] proposed a version of the theory of plasticity assuming that the yield surface moves as a rigid body. Later on Prager [2] independently discussed the same idea in application to the kinematic models interpreting the behavior of plastic systems. In [3,4] the possibility of analytic formulation of a law of anisotropic hardening has been studied as proposed by Ishlinskii and Prager. The present note studies the law of anisotropic hardening in the formulation of the paper [4].

It is shown that the version of the anisotropic hardening body proposed in [4] leads to equations of the hyperbolic type, extending the quantitative feature of the ideally plastic material to the case of hardening bodies. In this note relations, corresponding to the plane state of strain and likewise to three-dimensional problems, are considered. In the last case, it is assumed that the state of stress corresponds to an edge of the yield prism generalizing in accordance with [4] the Tresca plasticity condition.

The case of plane strain will be considered first. The basic relations may be given the form [4]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

$$[(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)]^2 + 4(\tau_{xy} - c\varepsilon_{xy})^2 = 4k^2, \qquad k,\ c = \text{const} \tag{2}$$

$$\frac{d\varepsilon_x}{(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)} = \frac{d\varepsilon_y}{(\sigma_y - c\varepsilon_y) - (\sigma_x - c\varepsilon_x)} = \frac{d\varepsilon_{xy}}{2(\tau_{xy} - c\varepsilon_{xy})} \tag{3}$$

The conditions (1)-(3) must be augmented by the conditions of compatibility of deformation

$$\frac{\partial \omega}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial x} = 0, \qquad \frac{\partial \omega}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y} = 0 \tag{4}$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \omega = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

For the following the notation $\epsilon_x = -\epsilon_y = \epsilon$, $\epsilon_{xy} = \gamma$ will be introduced. The condition (2) will be satisfied by writing

$$\sigma_x = p + k\cos 2\theta + c\varepsilon, \qquad \sigma_y = p - k\cos 2\theta - c\varepsilon, \quad \tau_{xy} = k\sin 2\theta + c\gamma \tag{5}$$

Substituting (5) into (1), one finds

$$\begin{gathered} \frac{\partial p}{\partial x} - 2k\sin 2\theta \frac{\partial \theta}{\partial x} + 2k\cos 2\theta \frac{\partial \theta}{\partial y} + c\frac{\partial \varepsilon}{\partial x} + c\frac{\partial \gamma}{\partial y} = 0 \\ \frac{\partial p}{\partial y} + 2k\cos 2\theta \frac{\partial \theta}{\partial x} + 2k\sin 2\theta \frac{\partial \theta}{\partial y} - c\frac{\partial \varepsilon}{\partial y} + c\frac{\partial \gamma}{\partial x} = 0 \end{gathered} \tag{6}$$

These relations must be supplemented by (3) and (4) which assume the forms

$$d\varepsilon \sin 2\theta - d\gamma \cos 2\theta = 0 \tag{7}$$

$$\frac{\partial \omega}{\partial x} - \frac{\partial \varepsilon}{\partial y} + \frac{\partial \gamma}{\partial x} = 0, \qquad \frac{\partial \omega}{\partial y} - \frac{\partial \varepsilon}{\partial x} - \frac{\partial \gamma}{\partial y} = 0 \tag{8}$$

It is readily verified that the five equations (6), (7) and (8) for the five unknowns p, θ, ϵ, γ, ω are of the hyperbolic type with the characteristics given by

$$dy - \operatorname{tg}\left(\theta \pm \tfrac{1}{4}\pi\right)dx = 0 \tag{9}$$

Let

$$d\xi = dy\cos\left(\theta + \tfrac{1}{4}\pi\right) - dx\sin\left(\theta + \tfrac{1}{4}\pi\right) \qquad d\eta = dy\cos\left(\theta - \tfrac{1}{4}\pi\right) - dx\sin\left(\theta - \tfrac{1}{4}\pi\right)$$

Along the characteristics one has relations, generalizing the well-known Hencky relations

$$\frac{\partial p}{\partial \xi} - 2k\frac{\partial \theta}{\partial \xi} - c\left(\frac{\partial \varepsilon}{\partial \eta}\cos 2\theta + \frac{\partial \gamma}{\partial \eta}\sin 2\theta\right) = 0 \quad \frac{\partial p}{\partial \eta} + 2k\frac{\partial \theta}{\partial \eta} - c\left(\frac{\partial \varepsilon}{\partial \xi}\cos 2\theta + \frac{\partial \gamma}{\partial \eta}\sin 2\theta\right) = 0 \tag{10}$$

It will be noted that $d\epsilon \cos 2\theta + d\gamma \sin 2\theta = d\gamma^*$, where $d\gamma^*$ is the shear strain along the characteristic.

It follows from Equation (7) that the relations of Geiringer apply, asserting the absence of extension along the characteristics.

Next, the three-dimensional problem will be considered. By [4] and [5] the condition of plasticity will be written

$$[(\sigma_x - c\varepsilon_x) - (\sigma - c\varepsilon) + 2/3k][\sigma_y - c\varepsilon_y) - (\sigma - c\varepsilon) + 2/3\,k] = (\tau_{xy} - c\varepsilon_{xy})^2 \quad (xyz) \tag{11}$$

where $\sigma = 1/3(\sigma_x + \sigma_y + \sigma_z)$, $\epsilon = 1/3(\epsilon_x + \epsilon_y + \epsilon_z)$. The symbol (xyz) denotes that the remaining relations are obtained by means of cyclic permutation of the subscripts.

Further, one has to use the transformation of variables

$$\sigma_x = p + 2k\cos^2\theta_x + c\varepsilon_x, \quad \tau_{xy} = 2k\cos\theta_x\cos\theta_y + c\varepsilon_{xy} \quad (xyz) \tag{12}$$

Substituting (12) in the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0 \qquad (xyz) \tag{13}$$

and supplementing them by the condition $\cos^2\theta_x = \cos^2\theta_y + \cos^2\theta_z = 1$, the relations of the law of plastic flow [6]

$$d\varepsilon_x + d\varepsilon_y + d\varepsilon_z = 0$$

$$\begin{aligned} d\varepsilon_x + d\varepsilon_{xy}\frac{\sigma_y - c\varepsilon_y - \sigma + 2/3\,k}{\tau_{xy} - c\varepsilon_{xy}} + d\varepsilon_{xz}\frac{\sigma_z - c\varepsilon_z - \sigma + 2/3\,k}{\tau_{xz} - c\varepsilon_{xz}} = \\ = d\varepsilon_{xy}\frac{\sigma_x - c\varepsilon_x - \sigma + 2/3\,k}{\tau_{xy} - c\varepsilon_{xy}} + d\varepsilon_y + d\varepsilon_{yz}\frac{\sigma_z - c\varepsilon_z - \sigma + 2/3\,k}{\tau_{yz} - c\varepsilon_{yz}} = \\ = d\varepsilon_{xz}\frac{\sigma_x - c\varepsilon_x - \sigma + 2/3\,k}{\tau_{xz} - c\varepsilon_{xz}} + d\varepsilon_{yz}\frac{\sigma_y - c\varepsilon_y - \sigma + 2/3\,k}{\tau_{yz} - c\varepsilon_{yz}} + d\varepsilon_z \end{aligned} \tag{14}$$

and the conditions of strain compatibility

$$\frac{\partial\omega_x}{\partial y} - \frac{\partial\varepsilon_y}{\partial z} + \frac{\partial\varepsilon_{yz}}{\partial y} = 0, \quad \frac{\partial\omega_x}{\partial z} + \frac{\partial\varepsilon_z}{\partial y} - \frac{\partial\varepsilon_{yz}}{\partial z} = 0 \quad (xyz) \tag{15}$$

one obtains a system of thirteen equations for the thirteen unknowns p, θ, ϵ_x, ϵ_{xy}, $\omega_x(xyz)$. It is readily verified that this system belongs to the hyperbolic type and that its characteristic determinant may be written in the form

$$(\operatorname{grad}\psi\mathbf{n})\,[(2\operatorname{grad}\psi\mathbf{n})^2 - (\operatorname{grad}\psi)^2] = 0 \tag{16}$$

where ψ is the equation of the characteristic surface, $\mathbf{n} = \cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{i} + \cos\theta_z\mathbf{k}$.

Thus, it has been shown that the generalization of the relations of the theory of ideal plasticity, proposed in [4], permits extension of the quantitative feature of the solutions of the theory of the ideally plastic body to the case of anisotropically hardening materials. In a certain manner this circumstance corresponds to experimental evidence: formation of Luder lines and slip surfaces.

It is known that the relations of the laws of the deformation theories of isotropic hardening reduce to equations of the elliptic type, complicated from the point of view of practical application; by the character of their assumptions the theories of isotropic hardening are of little use for the description of the actual behavior of plastic bodies, which is invariably accompanied by anisotropic hardening. The laws of the deformation theories of isotropic hardening essentially correspond in nature to isotropic nonlinearly elastic bodies.

The above results show that the simultaneous study of the effects of anisotropy and hardening may lead to a simplification of mathematical problems.

BIBLIOGRAPHY

1. Ishlinskii, A.Iu., Obshchaia teoriia plastichnosti s lineinym uprochneniem (The general theory of plasticity with linear hardening). *Ukr. Matemat. Zh.* Vol. 6, No. 3, 1954.

2. Prager, W., The theory of plasticity: a survey of recent achievements. *Proc. Inst. Mech. Eng.* 169, 41, 1955.

3. Kodashevich, Iu.I. and Novozhilov, V.V., Teoriia plastichnosti, uchityvaiushchaia mikronapiazheniia (Theory of plasticity, taking into account microstresses). *PMM* Vol. 22, No. 1, 1958.

4. Shield, R.T. and Ziegler, H., On Prager's hardening rule. *ZAMP* 9a, 1958.

5. Ivlev, D.D., Ob obshchikh uravneniiakh teorii ideal'noi plastichnosti i statiki sypuchei spedy (On the general equations of the theory of ideal plasticity and of statics of granular [pulverulent] media). *PMM* Vol. 22, No. 1, 1958.

6. Ivlev, D.D., O sootnosheniiakh, opredeliaiushchikh plasticheskoe techenie pri uslovii plastichnosti Treska i ego oboshcheniiakh (On the relations determining plastic flow for the plasticity condition of Tresca and generalizations). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 3, 1959.

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